Einstein spaces in warped geometries in five dimensions

M. Arık,* A. Baykal,† M. C. Çalık,‡ D. Çiftçi,§ and Ö. Delice‖
Department of Physics, Boğaziçi University, Bebek, Istanbul, Turkey
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We investigate five-dimensional Einstein spaces in warped geometries from the point of view of the four-dimensional physically relevant Robertson-Walker-Friedman cosmological metric and the Schwarzschild metric. We show that a four-dimensional cosmology with a closed spacelike section and a cosmological constant can be embedded into five-dimensional flat space-time.

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The general theory of relativity is an experimentally well-tested theory. Among these tests, the Schwarzschild solution has played a central role. For cosmological solutions, however, the situation is beginning to become clarified with the accumulation of relevant astrophysical data. On the one hand, a simple, consistent, logical cosmology requires a spatially maximally symmetric Robertson-Walker-Friedman cosmology with closed spacelike sections $(k=1)$. Recent observational evidence shows that we live in an expanding closed universe with a positive cosmological constant [1]. The maximally symmetric Einstein–de Sitter solutions are good prototypes of such space-times since they include the cosmological constant. However, the existence of the cosmological constant is one of the deep mysteries in cosmology.

Since the Kaluza-Klein idea [2], there have been many theories suggesting that the Universe may have more than four dimensions. Nowadays, the idea that our Universe may be a three-brane embedded in five-dimensional universe is very popular [3–5]. For a recent review see [6].

The recent interest in the Randall-Sundrum [4,5] and related scenarios has brought into consideration warped geometries such that the four-dimensional spacetime metric is multiplied by a warp factor which only depends on the coordinate of the extra dimension: namely,

$$ds^2_{(5)}=dw\otimes dw+b^2(w)\,\eta_{\mu\nu}dx^{\mu}\otimes dx^{\nu}, \tag{1}$$

where $b(w)=e^{-k|w|}$ is the warp factor, k is a constant and $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$. In their second scenario [5], where the range of the extra dimensions w is $-\infty<w<+\infty$, we live on a four-dimensional infinitely thin shell (three-brane). Notice that the five-dimensional Einstein tensor outside the brane satisfies the Einstein equation with a cosmological constant:

$${}^{(5)}G_{MN}+g_{MN}\Lambda_5=0,\qquad M,N=0,1,2,3,5 \tag{2}$$

and on $w=$const hypersurfaces 4-dimensional Einstein tensor of this metric satisfies

$${}^{(4)}G_{\mu\nu}+g_{\mu\nu}\Lambda_4=0,\qquad \mu,\nu=0,1,2,3, \tag{3}$$

where $\Lambda_5=-6k^2$ and $\Lambda_4=0$. The full Einstein tensor of the 5-dimensional space-time of the metric (1) is given by

$${}^{(5)}G_{MN}=-\eta_{MN}\Lambda_5-6k\,\delta^{\mu}_{M}\delta^{\nu}_{N}\eta_{\mu\nu}\delta(w). \tag{4}$$

Motivated by these considerations, in this work we will calculate five-dimensional Einstein equations of the metric (1) for arbitrary $b(w)$ in terms of the four-dimensional quantities originating from the four-dimensional metric

$$ds^2_{(4)}=g_{\mu\nu}dx^{\mu}\otimes dx^{\nu}, \tag{5}$$

and $b(w)$. As in the Randall-Sundrum scenario we do not wish any matter sources to survive on five-dimensional space-time except a possible five-dimensional cosmological constant. Our most important conclusion will be that a four-dimensional cosmological constant can be induced even when the five-dimensional cosmological constant is zero. We require that only gravity can propagate in extra dimensions. Thus the five-dimensional space-time is an Einstein space where the original Randall-Sundrum metric will be one of the cases of our solutions. Then, as in the Randall-Sundrum scenario we impose reflection (Z_2) symmetry on the extra dimension w. This symmetry will make the derivatives of the metric discontinuous with respect to w at the point of symmetry and we know from the thin shell formalism of general relativity [7] that this discontinuity will give rise to a surface layer (thin shell – brane). The resulting five-dimensional Einstein tensor will be of the form (4). Since in our solutions four-dimensional part of the metric is same for every w, the brane tension [the term proportional to $\delta(w)$] is caused only by the jump of $b'(w)$ on the brane.

After calculating the five-dimensional metric in terms of the four-dimensional metric, we first consider the four-dimensional cosmological solutions of Einstein equations where the four-dimensional space-time is an Einstein space and the four-dimensional hypersurface is devoid of matter except a four-dimensional cosmological constant. We tabulate all possible solutions we find in Table I.

We then consider the four-dimensional metric to be given by spherically symmetric static Schwarzschild solution. For this metric we also find all possible solutions $b(w)$ when

*Electronic address: arikm@boun.edu.tr
†Electronic address: baykala@boun.edu.tr
‡Electronic address: cem.calik@boun.edu.tr
§Electronic address: dciftci@boun.edu.tr
‖Electronic address: odelice@boun.edu.tr

TABLE I. $b(w), a(t), c(\chi)$ and other quantities for 5D Einstein space when 4D part is of the form (6).

k	$c(\chi)$	$a(t)$	$b(w)$	$R^{M(5)}_{NPQ}$	$R^{(5)}$	Λ_5	$R^{\mu(4)}_{\nu\lambda\kappa}$	$R^{(4)}$	Λ_4	$R^{(3)}$
0	χ	1	1	0	0	0	0	0	0	0
			$e^{b_o w}$	$-b_0^2$	$-20b_0^2$	$-6b_0^2$	0			
		$e^{a_0 t}$	$a_0 w$	0	0	0	a_0^2	$12a_0^2$	$3a_0^2$	
			$\frac{a_0}{b_0}\sinh(b_0 w)$	$-b_0^2$	$-20b_0^2$	$-6b_0^2$	a_0^2			
			$\frac{a_0}{b_0}\sin(b_0 w)$	b_0^2	$20b_0^2$	$6b_0^2$	a_0^2			
1	$\frac{1}{c_0}\sin(c_0\chi)$	$\frac{c_0}{a_0}\cosh(a_0 t)$	$a_0 w$	0	0	0	a_0^2	$12a_0^2$	$3a_0^2$	$6c_0^2$
			$\frac{a_0}{b_0}\sinh(b_0 w)$	$-b_0^2$	$-20b_0^2$	$-6b_0^2$	a_0^2			
			$\frac{a_0}{b_0}\sin(b_0 w)$	b_0^2	$20b_0^2$	$6b_0^2$	a_0^2			
-1	$\frac{1}{c_0}\sinh(c_0\chi)$	$c_0 t$	1	0	0	0	0	0	0	$-6c_0^2$
			$e^{b_0 w}$	$-b_0^2$	$-20b_0^2$	$-6b_0^2$	0			
		$\frac{c_0}{a_0}\sinh(a_0 t)$	$a_0 w$	0	0	0	a_0^2	$12a_0^2$	$3a_0^2$	
			$\frac{a_0}{b_0}\sinh(b_0 w)$	$-b_0^2$	$-20b_0^2$	$-6b_0^2$	a_0^2			
			$\frac{a_0}{b_0}\sin(b_0 w)$	b_0^2	$20b_0^2$	$6b_0^2$	a_0^2			
		$\frac{c_0}{a_0}\sin(a_0 t)$	$\frac{a_0}{b_0}\cosh(b_0 w)$	$-b_0^2$	$-20b_0^2$	$-6b_0^2$	$-a_0^2$	$-12a_0^2$	$-3a_0^2$	

five-dimensional metric is an Einstein space and collect them in Table II. Then we will also make some comments on these solutions.

Our five-dimensional metric ansatz can be written in an orthonormal basis as

$$ds^2_{(5)} = dw\otimes dw + b(w)^2\times\{g_{\mu\nu}(x^\rho)dx^\mu\otimes dx^\nu\}, \qquad (6)$$

$$= \eta_{AB}E^A\otimes E^B, \qquad (7)$$

where the four-dimensional metric is also written in an orthonormal basis:

$$ds^2_{(4)} = g_{ij}dx^i\otimes dx^j = \eta_{ij}e^i\otimes e^j. \qquad (8)$$

The orthonormal basis one forms are chosen as

$$E^i = b(w)e^i, \quad E^4 = ibe^4 = ibdt, \quad E^5 = dw. \qquad (9)$$

Note that for the sake of computational simplicity, we chose the timelike one form imaginary so that we can take η_{AB} as δ_{AB} and η_{ij} as δ_{ij}. The indices run as $A,B,\ldots = 1,2,3,4,5$ and $i,j = 1,2,3,4$.

Employing Cartan structure equations, we find the nonzero components of the Riemann tensor as follows:

TABLE II. $b(w)$ for different signs of b_0.

Λ_4	$b(w)$	$R^{(5)}$	Λ_5	$R^{(4)}$
0	1	0	0	0
	$e^{b_0 w}$	$-20b_0^2$	$-6b_0^2$	
$3d_0$	w	0	0	$12d_0^2$
	$\frac{d_0}{b_0}\sinh(b_0 w)$	$-20b_0^2$	$-6b_0^2$	
	$\frac{d_0}{b_0}\sin(b_0 w)$	$20b_0^2$	$6b_0^2$	
$-3d_0$	$\frac{d_0}{b_0}\cosh(b_0 w)$	$-20b_0^2$	$-6b_0^2$	$-12d_0^2$

$$ {}^{(5)}R^{ij}_{\ \ kl}=\frac{{}^{(4)}R^{ij}_{\ \ kl}}{b^2}-\delta^{ij}_{kl}\frac{b'^2}{b^2},\quad {}^{(4)}R^{i5}_{\ \ i5}=-\frac{b''}{b}, \tag{10}$$

where the $'$ on the functions denote derivatives of the functions with respect to their arguments, and δ^{ij}_{kl} is generalized Kronecker delta. The Ricci curvature scalar is found as

$$ {}^{(5)}R=\frac{{}^{(4)}R}{b^2}-8\frac{b''}{b}-12\frac{b'^2}{b^2}, \tag{11}$$

Using these one can easily calculate the nonzero components of the Einstein tensor $G^{(5)}_{AB}$ of the metric (6) as

$$ {}^{(5)}G_{ij}=\frac{{}^{(4)}G_{ij}}{b^2}+\delta_{ij}\left\{3\frac{b''}{b}+3\frac{b'^2}{b^2}\right\}, \tag{12}$$

$$ {}^{(5)}G_{55}=-\frac{R^{(4)}}{2b^2}+\frac{6b'^2}{b^2}. \tag{13}$$

We have calculated the nonzero components of the five-dimensional Einstein tensor for the metric (6) in terms of $b(w)$ and the four-dimensional Einstein tensor of the metric (8). Since we want to first investigate the cosmological solutions we chose four-dimensional metric ansatz as follows:

$$ ds^2_{(4)}=-dt^2+a(t)^2 ds_3^2, \tag{14}$$

where

$$ ds^2_{(3)}=d\chi^2+c(\chi)^2 d\Omega_2^2,\quad d\Omega_2^2=d\theta^2+\sin^2\theta d\phi^2. \tag{15}$$

Here we will find admissible values of $b(w),a(t),c(\chi)$ when the Einstein equations satisfy Eqs. (2) and (3). We can read off the orthonormal basis one forms e^i from Eqs. (14) and (15):

$$ e^i=\{e^4,a(t)e^a\},\quad e^4=i\,dt,$$

$$ e^a=\{d\chi,\ c(\chi)d\theta,\ c(\chi)\sin\theta d\phi\},\quad a,b\ \ldots=1,2,3.$$

For Eq. (10) the nonzero components of the four-dimensional Riemann tensor are found as

$$ {}^{(4)}R^{ab}_{\ \ cd}=\frac{{}^{(3)}R^{ab}_{\ \ cd}}{a^2}+\delta^{ab}_{cd}\frac{\dot{a}^2}{a^2},\quad {}^{(4)}R^{a4}_{\ \ a4}=\frac{\ddot{a}}{a}, \tag{16}$$

and

$$ {}^{(3)}R^{12}_{\ \ 12}={}^{(3)}R^{13}_{\ \ 13}=-\frac{\check{\check{c}}}{c},\quad {}^{(3)}R^{23}_{\ \ 23}=\frac{1-\check{c}^2}{c}. \tag{17}$$

For the Ricci curvature scalar (11), we have

$$ {}^{(4)}R=\frac{{}^{(3)}R}{a^2}+6\left\{\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right\},\quad {}^{(3)}R=-4\frac{\check{\check{c}}}{c}+2\frac{1-\check{c}^2}{c^2}. \tag{18}$$

The Einstein tensor for this metric (14) is

$$ {}^{(4)}G_{ab}=\frac{{}^{(3)}G_{ab}}{a^2}-\delta_{ab}\left\{\frac{2\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right\}, \tag{19}$$

$$ {}^{(4)}G_{44}=-\frac{R^{(3)}}{2a^2}-3\frac{\dot{a}^2}{a^2}, \tag{20}$$

where

$$ {}^{(3)}G_{11}=\frac{\check{c}^2-1}{c^2},\quad {}^{(3)}G_{22}={}^{(3)}G_{33}=\frac{\check{\check{c}}}{c}. \tag{21}$$

Let us combine all these, then ${}^{(5)}G_{AB}$ becomes

$$ {}^{(5)}G_{11}=\left\{\frac{\check{c}^2-1}{a^2c^2}-\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right\}\frac{1}{b^2}+\frac{3b''}{b}+\frac{3b'^2}{b^2}, \tag{22}$$

$$ {}^{(5)}G_{22}=\left\{\frac{\check{\check{c}}}{a^2c}-\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right\}\frac{1}{b^2}+\frac{3b''}{b}+\frac{3b'^2}{b^2} = {}^{(5)}G_{33}, \tag{23}$$

$$ {}^{(5)}G_{44}=\left\{\frac{2\check{\check{c}}}{a^2c}+\frac{\check{c}^2-1}{a^2c^2}-3\frac{\dot{a}^2}{a^2}\right\}\frac{1}{b^2}+\frac{3b''}{b}+\frac{3b'^2}{b^2}, \tag{24}$$

$$ {}^{(5)}G_{55}=\left\{\frac{2\check{\check{c}}}{a^2c}+\frac{\check{c}^2-1}{a^2c^2}-3\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right\}\frac{1}{b^2}+\frac{6b'^2}{b^2}. \tag{25}$$

As we said before, we want to solve these for a and b from ${}^{(5)}G_{AB}+\delta_{AB}\Lambda_5=0$. For ${}^{(5)}G_{11}={}^{(5)}G_{22}$ we get the following differential equation:

$$\frac{\check{\check{c}}}{c}=\frac{\check{c}^2-1}{c^2} \tag{26}$$

whose set of solutions is

$$c(\chi)=\left\{\chi,\frac{1}{c_0}\sin(c_0\chi),\frac{1}{c_0}\sinh(c_0\chi)\right\}, \tag{27}$$

which correspond respectively to the cases $k=0,1,-1$.

For $k=0$, ${}^{(5)}G_{ii}={}^{(5)}G_{44}$ gives the following differential equation:

$$\frac{\ddot{a}}{a}=\frac{\dot{a}^2}{a^2}, \tag{28}$$

whose set of solutions is

$$a(t)=\{1;e^{a_0t}\}. \tag{29}$$

For the $a=1$ case, ${}^{(5)}G_{ii}={}^{(5)}G_{55}$ gives the equation

$$\frac{b''}{b}=\frac{b'^2}{b^2},$$

whose set of solutions is

$$b(w)=\{1;e^{b_0w}\}.$$

Finally, for the $a=e^{a_0t}$ case, we have

$$\frac{b''}{b}=\frac{b'^2-a_0^2}{b^2},$$

whose set of solutions is

$$b(w)=\left\{a_0w;\frac{a_0}{b_0}\sinh(b_0w);\frac{a_0}{b_0}\sin(b_0w)\right\}.$$

In the same way, we can easily find the solutions for $k=\mp 1$. All solutions are shown in Table I. As in the Randall-Sundrum case, to have a brane embedded in five dimensions for these solutions we have to impose Z_2 symmetry on $b(w)$. Then our four-dimensional universe will be an infinitely thin shell at $w=0$ and the total five-dimensional Einstein tensor will have of the form:

$$\begin{aligned}{}^{(5)}_{(T)}G_{AB}&={}^{(5)}G_{AB}+6\frac{b'}{b}\,\delta(w)\\ &=-\Lambda_5\delta_{AB}-\sigma\delta_{AB}\delta^A_\mu\delta^B_\nu\delta(w).\end{aligned} \tag{30}$$

The $k=1$ case corresponds to closed expanding universe with positive cosmological constant, which is in accordance with recent observations [1]. For this case, Table I shows that $b(w)$ can take three different values: $\{w;\sinh w;\sin w\}$. The first of these is very interesting since in this case the five-dimensional Riemann tensor and the five-dimensional cosmological constant are zero.

To have a $k=1$ solution with ${}^{(5)}R^{MN}_{PQ}=0$ for this geometry, it is necessary to have nonzero Λ_4. So, flat and empty five-dimensional Minkowski universe in warped geometry (6) can give rise to a four-dimensional closed expanding universe with positive cosmological constant. Imposing Z_2 symmetry, the metric for this case becomes

$$\begin{aligned}ds^2_{(5)}=dw^2+(a_0|w|)^2\Bigg\{&-dt^2+\frac{c_0^2}{a_0^2}\cosh^2(a_0t)\\ &\times\left\{d\chi^2+\frac{1}{c_0^2}\sin^2(c_0\chi)d\Omega_2^2\right\}\Bigg\},\end{aligned} \tag{31}$$

with

$$\begin{aligned}&{}^{(5)}G_{\mu\nu}=-6b'/b\,\delta_{\mu\nu}\delta(w),\\ &{}^{(5)}G_{55}={}^{(5)}G_{5\mu}=0,\quad {}^{(4)}G_{\mu\nu}=-\Lambda_4\delta_{\mu\nu}.\end{aligned} \tag{32}$$

For this case the matter content of the four-dimensional universe is only the four-dimensional cosmological constant. In fact, observations show that, the cosmological constant dominates the matter content of the Universe. According to the recent review [8], the composition of the content of the Universe is as follows:

$$\begin{aligned}&\Omega_B\approx(0.01-0.2),\quad \Omega_R\approx 2\times10^{-5},\\ &\Omega_{DM}\approx 0.3,\quad \Omega_\Lambda\approx 0.7,\end{aligned} \tag{33}$$

where Ω_B is the density parameter of the visible, nonrelativistic, baryonic matter; Ω_R is the density parameter of the radiation; Ω_{DM} is the density parameter of the pressureless nonbaryonic dark matter; and Ω_Λ is the density parameter of the cosmological constant. According to the observations which use several independent techniques, the density parameter of the nonrelativistic matter is $\Omega_{NR}=(\Omega_B+\Omega_{(DM)})\approx(0.2\text{–}0.4)$. This raises the possibility whether with just a four-dimensional cosmological constant the five-dimensional space-time is flat except on the brane. Other kinds of matter in four dimensions require the five-dimensional space-time to fluctuate from flat. Thus the presence of five dimensions differentiates between ‘‘dark energy’’ satisfying equation of state $p=-\rho$ and other forms of matter-energy. Although four-dimensional de Sitter space with positive cosmological constant is consistent with five-dimensional flat space-time, other types of matter-energy in four dimensions require the five-dimensional space-time to fluctuate from flatness.

Now we turn to discuss the four-dimensional Schwarzschild solution from the five-dimensional point of view. Let us choose $ds^2_{(4)}$ as Schwarzschild–de Sitter metric which satisfies Eq. (3) and is given by

$$ds^2_{(4)} = -\left\{1 - \frac{2m}{r} - d_0 r^2\right\} dt^2 + \left\{1 - \frac{2m}{r} - d_0 r^2\right\}^{-1} dr^2 + r^2 d\Omega_2^2. \tag{34}$$

We find $b(w)$ for $d_0 < 0, d_0 > 0, d_0 = 0$ when the metric (6) satisfies Eq. (2) and presented in the Table II. Note that for $m = 0$ and for $d_0 > 0$, the Schwarzschild–de Sitter metric becomes a maximally symmetric metric and this metric can be written in a form where spacelike sections are closed. The metric (34) can be transformed into

$$ds^2_{(4)} = -dt'^2 + \cosh^2(t')\{d\chi^2 + \sin^2\chi d\Omega_2^2\} \tag{35}$$

with the following transformation:

$$r = \cosh(t')\sin(\chi),$$

$$t = \ln\left\{\frac{\sinh(t') + \cosh(t')\cos(\chi)}{\{1 - \cosh^2(t')\sin^2(\chi)\}^{1/2}}\right\}. \tag{36}$$

For this Schwarzschild–de Sitter case, for $b(w) \sim w$ and $m \neq 0$, five-dimensional Riemann tensor is not zero or constant but involves terms proportional to m/r^3. If $m = 0$, the solution reduces to Eq. (31). Thus, if we impose Z_2 symmetry, there will be a brane at $w = 0$. Having matter sources on the brane will change the five-dimensional metric from flat to curved. Five-dimensional Ricci flat but curved metric in warped geometry can give rise to a four-dimensional universe with positive cosmological constant and matter. This is a special case of space-time matter (or induced matter) theorem [9] which states that the matter content of the universe is induced from higher-dimensional geometry. The relevance of this theorem has been emphasized from the RS point of view by Wesson and Seahra [10].

In conclusion, we have shown that if in a Randall-Sundrum like scenario one imposes the condition that (4+1)-dimensional space-time is flat, the only (3+1)-dimensional brane which admits a closed spacelike section cosmology requires a four-dimensional cosmological constant. It is clear from Table I that in fact all flat five-dimensional space-time manifolds in warped geometries (6) imply a nonzero and positive cosmological constant for the four-dimensional cosmology. This fact may be important as far as the measured [1] cosmological constant is positive.

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